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Degree of Approximation of Functions by Hausdorff Mean of Their Fourier Series in the Hölder Metric

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Abstract

The main purpose of the present paper is to study the degree of approximation of functions by Housdorff means of their Fourier series generalizing some known results in the literature.

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1. Definitions and Notations

The sequence $\{\mu_n\}$ is said to be a moment sequence if

$$\mu_n = \int_0^1 \mu^n d\chi(u) \qquad (n = 0, 1, 2, ...) \tag{1}$$

where $\chi(\mu)$ is called the mass function of moments μ_n and is of bounded variation in the closed interval $[0, 1]$. It is also supposed that $\chi(0) = 0$ and $\mu_0 = \int_0^1 d\chi(u) = 1$. The conditions for moment sequence imply $\chi(1) = 1$.

Further if $\chi(u)$ is continuous at the origin; that is

$$\chi(0^+) = \chi(0) = 0$$

then $\chi(u)$ is called a regular mass function and $\{\mu_n\}$ is called regular moment sequence. Let $\sum_{n=0}^{\infty} a_n$ be an infinite series and $\{s_n\}$ be the sequence of its partial sums. Corresponding to a moment sequence $\{\mu_n\}$ or a mass function $\chi(u)$, we write the sequence to sequence transformation by

$$t_n = \sum_{k=0}^{n} \left\{ \int_0^1 \binom{n}{k} u^k (1-u)^{n-k} d\chi(u) \right\} s_k \tag{2}$$

OR

$$t_n = \sum_{k=0}^{n} \binom{n}{k} (\Delta^{n-k} \mu_k) s_k \tag{3}$$

where for $n \geq 0, \Delta^0 \mu_n = \mu_n; \Delta^P \mu_n = \Delta^{P-1}(\mu_n - \mu_{n+1}), P \geq 1.$

The sequence $\{s_n\}$ (or the series $\sum_{n=0}^{\infty} a_n$) is said to be Hausdorff summable to s [4], [8] if $\lim_{n\to\infty} t_n = s$; sequence $\{t_n\}$ is called Hausdorff mean of sequence $\{s_n\}$.

Let $\mu = (\mu_{nk})$ and $\delta = (\delta_{mn})$ be triangular matrices defined respectively by

$$\mu_{nn} = \mu_n (n = 0, 1, 2, ...)$$

$$\mu_{nk} = 0 \quad (n \neq k)$$

$$\text{and } \delta_{mn} = \begin{cases} (-1)^n \binom{m}{n}; n \leq m \\ 0, n > m \end{cases} \tag{4}$$

Then the matrix $A = \delta\mu\delta$ is called a Hausdorff matrix, if $\{\mu_n\}$ is a moment sequence (or if $\chi(u)$ is a mass function). Thus the Hausdorff matrix $A = (a_{nk})$ are given by

$$a_{nk} = \begin{cases} \sum_{m=k}^{n} \delta_{nm}.\mu_m.\delta_{mk}, & k \leq n \\ 0, & k > n \end{cases} \tag{5}$$

whence by use of (1.1) we get

$$a_{nk} = \begin{cases} \int_0^1 \binom{n}{k} u^k (1-u)^{n-k}.d\chi(u), & k \leq n \\ 0, & k > n \end{cases} \tag{6}$$

It is easily seen that if the mass function $\chi(u)$ is continuous at the origin then the Hausdorff matrix is regular. if the mass function $\chi(u) = \{1-(1-u)^{\alpha}\}, \alpha > 0, 0 \leq u \leq 1$, then $\mu_n = \binom{n+\alpha}{n}^{-1}$ and Hausdorff method reduces to familiar Cesaro (C, α) method (see Hardy [4]). On the otherhand, if we take for $q > 0$

$$\chi(u) = \begin{cases} 0, & \text{when } 0 \leq u < \frac{1}{1+q} \\ 1, & \text{when } \frac{1}{1+q} \leq u \leq 1 \end{cases}$$

then Hausdorff method reduces to familiar Euler's method (see Hardy[4])

Let $f(t)$ be a periodic function of period 2π and integrable in the sense of Lebesgue over $[-\pi, \pi]$. Let the Fourier series of f at $t = x$ be given by

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{7}$$

and $S_n(x)$ be the sequence of partial sums of the series (1.7). We write

$$D_n(t) = \frac{\sin\left(n + \frac{1}{2}\right) t}{2 \sin t/2} \tag{8}$$

$$\varphi_x(t) = \{f(x+t) + f(x-t) - 2f(x)\} \tag{9}$$

It is easily seen that

$$S_n(x) - f(x) = \frac{1}{\pi} \int_0^{\pi} \varphi_x(t).D_n(t)dt \tag{10}$$

As we are concerned with degree of approximation of functions by the Hausdorff mean of their Fourier series we write using(1.2) the Hausdorff mean of the sequence

$$\{S_n(x)\} \text{ as } H_n(f(x), \chi) = \sum_{k=0}^{n} \left\{ \int_0^1 \binom{n}{k} u^k (1-u)^{n-k}.d\chi(u) \right\} S_k(x) \tag{11}$$

using(1.10)

$$H_n(f(x),\chi) = \sum_{k=0}^{n}\left\{\int_0^1 \binom{n}{k} u^k(1-u)^{n-k}.d\chi(u)\right\}\left\{\frac{1}{\pi}\int_0^\pi \varphi_x(t)D_k(t)dt + f(x)\right\}$$
$$= \sum_{k=0}^{n}\left\{\int_0^1 \binom{n}{k} u^k(1-u)^{n-k}.d\chi(u)\right\}\left\{\frac{1}{\pi}\int_0^\pi \varphi_x(t)D_k(t)dt\right\} + f(x)$$

whence using the fact that

$$\sum_{k=0}^{n}\int_0^1 \binom{n}{k} u^k(1-u)^{n-k}d\chi(u) = 1$$

We get

$$H_n(f(x),\chi) - f(x) = \int_0^\pi \varphi_x(t)\left[\sum_{k=0}^{n}\int_0^1 \binom{n}{k} u^k(1-u)^{n-k}d\chi(u).\frac{\sin\left(k+\frac{1}{2}\right)t}{2\pi\sin t/2}\right]dt$$
$$= \int_0^\pi \varphi_x(t)\left[\sum_{k=0}^{n}\binom{n}{k}\frac{\sin\left(k+\frac{1}{2}\right)t}{2\pi\sin t/2}\int_0^1 u^k(1-u)^{n-k}.d\chi(u)\right]dt$$
$$= \int_0^\pi \varphi_x(t)H_n(t)dt \qquad (12)$$

where

$$H_n(t) = \sum_{k=0}^{n}\binom{n}{k}\frac{\sin\left(k+\frac{1}{2}\right)t}{2\pi\sin t/2}\int_0^1 u^k(1-u)^{n-k}.d\chi(u) \qquad (13)$$

$H_n(t)$ is called the Hausdorff kernel.

By simple Computation we obtain from (1.13).

$$H_n(t) = \frac{1}{2\pi}\int_0^1 R^n(u,t)\cos n\Theta d\chi(u) + \frac{1}{2\pi}\cot\frac{t}{2}\int_0^1 R^n(u,t)\sin\ n\Theta d\chi(u) \qquad (14)$$

where $R(u,t) = |1-u+ue^{it}|$

$$\Theta = \tan^{-1}\frac{u\sin t}{1-u+u\cos t}$$

Again writing $Q_n^r(t) = \frac{1}{2\pi}\int_0^1 R^n(u,t)\cos n\Theta d\chi(u)$ and

$$Q_n^i(t) = \frac{1}{2\pi}\int_0^1 R^n(u,t)\sin n\Theta d\chi(u)$$

We obtain from (1.14)

$$H_n(t) = Q_n^r(t) + Q_n^i(t)\cot t/2 \tag{15}$$

Further it is easy to see that

$$(i)\ H_n(t) \text{ is an even function}$$

and

$$(ii)\ \int_{-\pi}^{\pi} H_n(t)dt = 1 \tag{16}$$

Let $C_{2\pi}$ denote the Banach space of all 2π periodic continuous functions under sup-norm. For $0 < \alpha \leq 1$ and some positive constant K, the function space H_α is given by

$$H_\alpha = \{f \in C_{2\pi} : |f(x) - f(y)| \leq K|x-y|^\alpha\} \tag{17}$$

The space H_α is a Banach space [9] with norm $\|.\|_\alpha$ defined by

$$\|f\|_\alpha = \|f\|_c + \sup_{x\neq y} \Delta^\alpha(f(x,y)) \tag{18}$$

where $\|f\|_c = \sup_{-\pi\leq x\leq\pi} |f(x)|$
and $\Delta^\alpha(f(x,y)) = \dfrac{|f(x)-f(y)|}{|x-y|^\alpha}, (x \neq y)$.
We shall use the convention that $\Delta^0 f(x,y) = 0$. The metric induced by the norm (1.18) is called a Hölder metric. It can be seen that

$$\|f\|_\beta \leq (2\pi)^{\alpha-\beta}\|f\|_\alpha, (0 \leq \beta < \alpha \leq 1) \tag{19}$$

Thus $\{H_\alpha, \|.\|_\alpha\}$ is a family of Banach spaces which decreases as α increases that is

$$C_{2\pi} \supseteq H_\beta \supseteq H_\alpha \qquad (0 \leq \beta < \alpha \leq 1)$$

The space $L_p[0,2\pi]$ when $\rho = \infty$ includes the space $C_{2\pi}$ of all continuous functions defined over $[0,2\pi]$.

We write

$$\|f\|_p = \begin{cases} \left(\frac{1}{2\pi}\int_0^{2\pi}|f(t)|^p.dt\right)^{\frac{1}{p}}, & p \geq 1 \\ \int_0^{2\pi}|f(t)|^p.dt, & 0 < p < 1 \\ \|f\|_c, & p = \infty \end{cases} \tag{20}$$

and

$$w(\delta) = w(\delta, f) = \sup_{0\leq h\leq\delta} \|f(x+h) - f(x)\|_c \tag{21}$$

$$w_p(\delta) = w_p(\delta, f) = \sup_{0\leq h\leq\delta} \|f(x+h) - f(x)\|_p \tag{22}$$

$$w_p^2(\delta) = w_p^2(\delta, f) = \sup_{0\leq h\leq\delta} \|f(x+h) + f(x-h) - 2f(x)\|_p \tag{23}$$

which are respectively called modulus of continuity, integral modulus of continuity and integral modulus of smoothness ([14], p.42).

In the case $0 < \alpha \leq 1$ and $w_p(\delta, f) = 0(\delta^\alpha)$. We write $f \in Lip(\alpha, p)$. The case $\alpha > 1$ is of no interest as in this case f turns out to be constant. The class $Lip(\alpha, p)$ with $P = \infty$ will be taken as $Lip\ \alpha$.

Hölder metric has been generalized in [3] as follows.

For $0 < \alpha \leq 1$ write

$$H(\alpha, p) = \{f \in L_p, 0 < p \leq \infty : \|f(x+h) - f(x)\|_p \leq k|h|^\alpha\}.$$

and define for $f \in H(\alpha, p)$

$$\|f\|_{(\alpha,p)} = \|f\|_p + \sup_{h\neq 0} \frac{\|f(x+h) - f(x)\|_p}{|h|^\alpha} \tag{24}$$

$$\|f\|_{(0,p)} = \|f\|_p.$$

It can be easily verified that (1.24) is a norm for $p \geq 1$ and a p-norm in the case $0 < p < 1$. Note that $H(\alpha, \infty)$ is the familiar H_α space introduced earlier by Prössdorf [9].

We write

$$E_n(u,t) = e^{cnu(1-u)t^2}, (0 \le u \le 1), (0 \le t \le \pi) \tag{25}$$

$$l_n(x) = H_n(f(x), \chi) - f(x) \tag{26}$$

$$F(t) = \varphi_{x+y}(t) - \varphi_x(t) \tag{27}$$

$f \uparrow$ to denote f is non decreasing

$f \downarrow$ to denote f is non increasing .

2. Introduction and Statement of the Theorem

Prössdorff [9] studied the degree of approximation in the Hölder metric and proved the following theorem.

Theorem A[9] Let $f \in H_\alpha(0 < \alpha \le 1)$ and $0 \le \beta < \alpha \le 1$

Then

$$\|\delta_n(f) - f\|_\beta = O(1) \begin{cases} n^{\beta-\alpha}, & 0 < \alpha < 1 \\ n^{\beta-1}(1+\log n)^{1-\beta}, & \alpha = 1 \end{cases}$$

where $\delta_n(f)$ is the Fejer mean of the Fourier series of f.

The case $\beta = 0$ is due to Alexists[1]. Chandra [2] obtained a generalization of Theorem A in the Nörlund or (N, p_n) transform and Reiesz transform set up with regard to approximation of functions in L_p norm the following theorem is due to Quade.

Theorem B [10] Let $f \in Lip(\alpha, p), (0 < \alpha \le 1)$. Then

$$\|\delta_n(f) - f\|_p = O(1) \begin{cases} n^{-\alpha}, & (p > 1) \\ n^{-\alpha}, & (p = 1, 0 < \alpha < 1) \\ \frac{\log n}{n}, & (p = 1, \alpha = 1) \end{cases}$$

With a view to generalise the above results in Nörlund transformation set up attempts were made by Sahaney and Rao[12], Chandra [2], Khan[6]. Mohapatra and Russel [7]. Considered this in generalized Nörlund means set up. With regard to the approximation of function in the generalised Hölder metric by matrix mean see Das, Ghosh and Ray[3]. In 2001, Rhodes has studied the degree of approximation of functions belonging to a certain weighted class by their Fourier series using Hausdorff means possessing mass function with bounded derivatives.

The objective of the present paper is to study the approximation problems by Hausdorff mean in the generalised Hölder metric.

We prove the following.

Theorem Let

(i) $\chi(u)$ be absolutely continuous over interval $(0, 1)$.

(ii) $\chi'(u)$ be positive and non decreasing in $(0, 1)$

(iii) $f \in H(\alpha, p)$ and $p \geq 1$.

Then for $0 \leq \beta < \alpha \leq 1$

$$\|H_n(f(x), \chi) - f(x)\|_{(\beta,p)} = O(1)\frac{1}{n^{\alpha-\beta}}\left[\int_{\frac{1}{n}}^{1} \frac{\chi(z).dz}{z^{\alpha-\beta+1}} + \int_{\frac{1}{n}}^{1} \frac{\chi(1) - \chi(1-z)}{z^{\alpha-\beta+1}}dz\right] + O(1)\frac{1}{n^{\alpha-\beta}}\int_{\frac{1}{n}}^{1-\frac{1}{n}} \frac{\chi'(u)du}{[u(1-u)]^{\alpha-\beta}} \tag{28}$$

3. Lemmas

To prove the theorem we use the following Lemmas.

Lemma 1 Let $l_n(x) = H_n(f(x), \chi) - f(x)$. If $f \in H(\alpha, p), p \geq 1$

Then

$$\|l_n(x + y) - l_n(x)\|_p \leq \begin{cases} \int_0^{\pi} \|\varphi_{x+y} - \varphi_x\|_p |H_n(t)| dt, & p \geq 1 \\ \int_0^{\pi} \|\varphi_{x+y} - \varphi_x\|_p |H_n(t)|^p dt, & 0 < p < 1 \end{cases}$$

Proof

Using (1.12) and (1.20) for $p \geq 1$

$$l_n(x) = \int_0^{\pi} \varphi_x(t).H_n(t).dt$$

and

$$\|l_n(x+y)-l_n(x)\|_p = \left[\frac{1}{\pi}\int_0^{\pi} |l_n(x+y)-l_n(x)|^p dx\right]^{\frac{1}{p}}$$
$$= \left[\frac{1}{\pi}\int_0^{\pi} dx \left|\int_0^{\pi} \{\varphi_{x+y}(t)-\varphi_x(t)\} H_n(t)dt\right|^p\right]^{\frac{1}{p}}$$

whence by generalised Minkowski inequality we obtain

$$\|l_n(x+y)-l_n(x)\|_p \leq \int_0^{\pi} \|\varphi_{x+y}-\varphi_x\|_p |H_n(t)|dt$$

In the case $0 < p < 1$, we use modified generalised Minkowski inequality and obtain the result.

This completes the proof.

Lemma 2 Let $f \in H(\alpha, p), 0 < \alpha \leq 1, 0 < p \leq \infty$. Then

$$\|\varphi_{x+y}(t)-\varphi_x(t)\|_p = O(1)\left\{\begin{array}{l}|t|^{\alpha}\\ |y|^{\alpha}.\end{array}\right.$$

Proof For $P \geq 1$ and using Minkowski's inequality

$$\left(\int_0^{2\pi} |\varphi_x(t)|^p.dx\right)^{\frac{1}{p}} \leq \left(\int_0^{2\pi} |f(x+t)-f(x)|^p dx\right)^{\frac{1}{p}} + \left(\int_0^{2\pi} |f(x-t)-f(x)|^p dx\right)^{\frac{1}{p}}.$$

and for $0 < p < 1$ we have the modified Minkowski type inequality

$$\int_0^{2\pi} |\varphi_x(t)|^p dx \leq \int_0^{2\pi} |f(x+t)-f(x)|^p.dx + \int_0^{2\pi} |f(x-t)-f(x)|^p dx.$$

Thus the first order estimate follows.

For proving the second order estimate we first note that

$$\varphi_{x+y}(t)-\varphi_x(t) = \{f(x+y+t)-f(x+t)\}+\{f(x+y-t)-f(x-t)\}-2\{f(x+y)-f(x)\}$$

and then apply Minkowski's inequality separately for $p \geq 1$ and for $0 < p < 1$.

Lemma 3 Let $f \in H(\alpha, p), 0 < \alpha \leq 1, 0 < p \leq \infty$. Then

$$\|F(t)\|_P = O(1)|y|^{\beta}.|t|^{\alpha-\beta}.$$

Proof Using (1.27) and Lemma 2, we have

$$\|F(t)\|_p = O(1)\left\{\begin{array}{l}|t|^{\alpha}\\ |y|^{\alpha}.\end{array}\right. \tag{29}$$

Writing $\|F(t)\|_p = \|F(t)\|_p^{\beta/\alpha}.\|F(t)\|_p^{1-\beta/\alpha}$ and using the estimates in (3.1) we obtain

$$\|F(t)\|_p = O(1)|y|^{\beta}.|t|^{\alpha-\beta}$$

This completes the proof.

Lemma 4 [5] For $0 \leq u \leq 1, 0 \leq t \leq \pi$

(i) $R(u,t) = O(1)e^{-cu(1-u)t^2}(c > 0)$

(ii) $R^n(u,t) = O(1)e^{-cnu(1-u)t^2}(c > 0)$

(iii) $H_n(t) = O(n)$

(iv) $H_n(t) = O(1)\frac{1}{t}\int_0^1 e^{-cnu(1-u)t^2}d\chi(u) \quad (c > 0)$

Lemma 5 [13] For small $t \in (0,\pi)$ and fixed δ_0

$$\Theta = ut + Au(1-u)t^3$$

where $|A| \leq \delta_0$.

Lemma 6 For small $t \in (0,\pi)$

(i) $\sin n\Theta - \sin nut = O(1)nu(1-u)t^3$.

(ii) $\cos n\Theta - \cos nut = O(1)nu(1-u)t^3$.

Proof of (i)

$|\sin n\Theta - \sin\ nut| = |2\cos\frac{n\Theta + nut}{2}, \sin\frac{n\Theta - nut}{2}| \leq n|\Theta - ut| = O(1)nu(1-t)t^3$ [by Lemma 5]

Hence the proof.

We omit the proof of (ii) as it can be proved by using arguments similar to those used in proving lemma 6(i).

Lemma 7 For $0 \leq \beta \leq 1$ and $0 \leq u \leq 1$

$$e^{-cnu(1-u)t^2} = O(1)\frac{1}{n^{\beta}}\left[\frac{1}{(u(1-u)t^2)^{\beta}}\right](c > 0)$$

Proof We have

$$En(u,t) = e^{cnu(1-u)t^2} \geq 1 \tag{30}$$

$$En(u,t) = e^{cnu(1-u)t^2} > cnu(1-u)t^2 \tag{31}$$

Writing $E_n(u,t) = [E_n(u,t)]^{1-\beta} . [E_n(u,t)]^{\beta}$ and using (3.2) and (3.3) we get

$$E_n(u,t) > [cnu(1-u)t^2]^{\beta} \tag{32}$$

$$\Rightarrow e^{-cnu(1-u)t^2} = \frac{1}{E_n(u,t)} = O(1)\frac{1}{n^{\beta}}\left[\frac{1}{(u(1-u)t^2)^{\beta}}\right].$$

This completes the proof of the Lemma.

Lemma 8 For $0 < \alpha \le 1, 0 \le u \le 1, k \in N, C > 0$ and $0 \le \beta < \alpha$

$$\int_{U_1}^{U_2} \theta^{\frac{k+\alpha-\beta}{2}}.e^{-c\theta}.d\theta = O(1)\begin{cases} n^{\frac{k+\alpha-\beta}{2}}.[u(1-u)]^{\frac{k+\alpha-\beta}{2}} \\ n^{\frac{k-\alpha+\beta}{2}}.[u(1-u)]^{\frac{k-\alpha+\beta}{2}} \end{cases}$$

where

$$U_1 = \pi^2 u(1-u).n^{-1}, U_2 = \delta^2 u(1-u)n$$

and δ is a fixed number between 0 and π .

Proof : We have

$$\begin{aligned}\int_{U_1}^{U_2} \theta^{\frac{k+\alpha-\beta}{2}}.e^{-c\theta}.d\theta &\le U_2^{\frac{k+\alpha-\beta}{2}}.\int_{U_1}^{U_2} e^{-c\theta}.d\theta \\ &\le U_2^{\frac{k+\alpha-\beta}{2}}.\int_{0}^{\infty} e^{-c\theta}.d\theta \\ &= O(1)U_2^{\frac{k+\alpha-\beta}{2}} \\ &= O(1)n^{\frac{k+\alpha-\beta}{2}}.[u(1-u)]^{\frac{k+\alpha-\beta}{2}} \end{aligned} \tag{33}$$

Again

$$\begin{aligned}\int_{U_1}^{U_2} \theta^{\frac{k+\alpha-\beta}{2}}.e^{-c\theta}.d\theta &= \int_{U_1}^{U_2} \theta^{\frac{k-\alpha+\beta}{2}}.\theta^{\alpha-\beta}.e^{-c\theta}.d\theta \\ &= U_2^{\frac{k-\alpha+\beta}{2}} \int_{U_1}^{\infty} .\theta^{\alpha-\beta}.e^{-c\theta}.d\theta \\ &= O(1)U_2^{\frac{k-\alpha+\beta}{2}} \\ &= O(1)n^{\frac{k-\alpha+\beta}{2}}.[u(1-u)]^{\frac{k-\alpha+\beta}{2}} \end{aligned} \tag{34}$$

Proof of the lemma follows from (3.5) and (3.6) .

Lemma 9 Let

(i) $\chi(u)$ be absolutely continuous over interval $(0,1)$.

(ii) $\chi^{'}(u)$ be positive in $(0,1)$.

Then for $0 \le \beta < \alpha \le 1$

$$(i)\ \chi\left(\frac{1}{n}\right) = O(1)\frac{1}{n^{\alpha-\beta}}\int_{\frac{1}{n}}^{1}\frac{\chi(y)}{y^{\alpha-\beta+1}}\ dy$$

$$(ii)\ \chi(1)-\chi\left(1-\frac{1}{n}\right) = O(1)\frac{1}{n^{\alpha-\beta}}\int_{\frac{1}{n}}^{1}\frac{\chi(1)-\chi(1-y)}{y^{\alpha-\beta+1}}\ dy$$

$$(iii)\ \frac{1}{n^{\alpha-\beta}}\int_{\frac{1}{n}}^{1}\frac{\chi^{1}(1-y)}{y^{\alpha-\beta}}\ dy = O(1)\frac{1}{n^{\alpha-\beta}}\int_{\frac{1}{n}}^{1}\frac{\chi(1)-\chi(1-y)}{y^{\alpha-\beta+1}}\ dy$$

Proof (i) We have

$$\begin{aligned}
&\frac{1}{n^{\alpha-\beta}}\int_{\frac{1}{n}}^{1}\frac{\chi^{1}(y)\ dy}{y^{\alpha-\beta}}\\
&= \frac{1}{n^{\alpha-\beta}}\left[\frac{\chi(y)}{y^{\alpha-\beta}}\right]_{\frac{1}{n}}^{1} + \frac{\alpha-\beta}{n^{\alpha-\beta}}\int_{\frac{1}{n}}^{1}\frac{\chi(y)dy}{y^{\alpha-\beta+1}}\\
&= \frac{1}{n^{\alpha-\beta}} - \chi\left(\frac{1}{n}\right) + \frac{\alpha-\beta}{n^{\alpha-\beta}}\int_{\frac{1}{n}}^{1}\frac{\chi(y)dy}{y^{\alpha-\beta+1}}
\end{aligned}$$

whence we get

$$\begin{aligned}
\chi\left(\frac{1}{n}\right) &= \frac{1}{n^{\alpha-\beta}} + \frac{\alpha-\beta}{n^{\alpha-\beta}}\int_{\frac{1}{n}}^{1}\frac{\chi(y)dy}{y^{\alpha-\beta+1}} - \frac{1}{n^{\alpha-\beta}}\int_{\frac{1}{n}}^{1}\frac{\chi^{1}(y)}{y^{\alpha-\beta}}dy\\
&= O(1)\frac{1}{n^{\alpha-\beta}}\int_{\frac{1}{n}}^{1}\frac{\chi(y)dy}{y^{\alpha-\beta+1}}dy
\end{aligned}$$

(since $\chi^{1}(u)$ is positive)

This completes the proof of Lemma 9(i) . We omit the proof of Lemma 9(ii) and Lemma 9 (iii) as it can be proved by using the relation

$$\frac{1}{n^{\alpha-\beta}}\int_{\frac{1}{n}}^{1}\frac{\chi^1(1-y)dy}{y^{\alpha-\beta}}$$
$$=\frac{1}{n^{\alpha-\beta}}+\frac{\alpha-\beta}{n^{\alpha-\beta}}\int_{\frac{1}{n}}^{1}\frac{\chi(1)-\chi(1-y)}{y^{\alpha-\beta+1}}\ dy$$
$$-\left[\chi(1)-\chi\left(1-\frac{1}{n}\right)\right]\ \text{and}$$

arguments similar to those used in proving Lemma 9 (i) .

Lemma 10 For fixed $t \in (0,\pi)$ and $u \in [0,1]$, Let

(a) $\chi(u)$ be absolutely continuous over interval $(0,1)$

(b) $\chi^1(u)$ be positive

Then

$$\int_0^1 e^{-cnu(1-u)t^2}.d\chi(u)$$
$$=O(1)\left[\chi\left(\frac{1}{n}\right)+\left\{\chi(1)-\chi\left(1-\frac{1}{n}\right)\right\}\right]$$
$$+O(1)\frac{1}{n^{\alpha-\beta}}\int_{\frac{1}{n}}^{1-\frac{1}{n}}\frac{\chi^1(u)du}{[u(1-u)]^{\alpha-\beta}}\quad (0\le\beta<\alpha\le 1, C>0)$$

Proof We write

$$\int_0^1 e^{-cnu(1-u)t^2}.d\chi(u)$$
$$=\left[\int_0^{\frac{1}{n}}+\int_{\frac{1}{n}}^{1-\frac{1}{n}}+\int_{1-\frac{1}{n}}^{1}\right]e^{-cnu(1-u)t^2}.d\chi(u)$$
$$=K_1+K_2+K_3\quad\text{(say)}$$

Using the fact that $e^{-cnu(1-u)t^2}\le 1$ for k_1 and k_3

We obtain

$$K_1\le\int_0^{\frac{1}{n}}d\chi(u)=O(1)\chi\left(\frac{1}{n}\right)$$
$$K_3\le\int_{1-\frac{1}{n}}^{\frac{1}{n}}d\chi(u)=O(1)\left[\chi(1)-\chi\left(1-\frac{1}{n}\right)\right]$$

Using Lemma 7 for K_2 , we obtain

$$K_2 = O(1)\frac{1}{n^{\alpha-\beta}}\int_{\frac{1}{n}}^{1-\frac{1}{n}} \frac{\chi^1(u)du}{[u(1-u)]^{\alpha-\beta}}$$

Collecting the estimates for K_1, K_3 and K_2 we obtain the result .

Lemma 11 For $t > \dfrac{\pi}{n}$ and $h = \dfrac{\pi}{nt}$, let

(a) $\chi(u)$ be absolutely continuous over interval $(0, 1)$.

(b) $\chi^1(u)$ be positive and $\chi^{'} \uparrow$ in $(0, 1)$

Then

$$\begin{aligned}(i)\ &\int_0^1 \chi^1(u)R^n(u,t)\sin nutdu\\ &= O(1)\left[\chi\left(\frac{\pi}{nt}\right) + \left\{\chi(1) - \chi\left(1-\frac{\pi}{nt}\right)\right\}\right]\\ &+ O(1)\left(\frac{\chi^1\left(1-\frac{\pi}{nt}\right)}{nt}\right)\end{aligned}$$

$$\begin{aligned}(ii)\ &\int_0^1 \chi^1(u)R^n(u,t)\cos nutdu\\ &= O(1)\left[\chi\left(\frac{\pi}{nt}\right) + \left\{\chi(1) - \chi\left(1-\frac{\pi}{nt}\right)\right\}\right]\\ &+ \frac{\chi^1\left(1-\frac{\pi}{nt}\right)}{nt}.\end{aligned}$$

Proof of (i) We write

$$\begin{aligned}(i)\ &\int_0^1 \chi^1(u)R^n(u,t)\sin nutdu\\ &= \left[\int_0^h + \int_h^{1-h} + \int_{1-h}^1\right]\chi^1(u)R^n(u,t)\sin nutdu\\ &= M_1 + M_2 + M_3 \quad \text{(say)}\end{aligned}$$

AS $|R^n(u,t)\sin nut| \le 1$, we getM_1 and M_3 we get

$$M_1 = O(1)\chi\left(\frac{\pi}{nt}\right)$$
$$\text{and}M_3 = O(1)\left[\chi(1) - \chi\left(1-\frac{\pi}{nt}\right)\right]$$

Again if $h = \dfrac{\pi}{nt} < \dfrac{1}{2}$ we write

$$\begin{aligned} M_2 &= \left[\int_h^{\frac{1}{2}} + \int_{\frac{1}{2}}^{1-h} \right] \chi^1(u) R^n(u,t) \sin nut \; du \\ &= M_{21} + M_{22} \qquad \text{(say)} \end{aligned}$$

Incase $h = \frac{\pi}{nt} > \frac{1}{2}$, we need not split the integral M_2 .

Case I For $h = \dfrac{\pi}{nt} < \dfrac{1}{2}$ we have for $h < \xi < \eta < \dfrac{1}{2}$

$$\begin{aligned} M_{21} &= \int_h^{\frac{1}{2}} \chi^1(u) R^n(u,t) \sin nut du \\ &= \chi^1 \left(\frac{1}{2} \right) \int_{\xi}^{\frac{1}{2}} R^n(u,t) \sin nut du \\ &= \chi^1 \left(\frac{1}{2} \right) R^n(\xi, t) \int_{\xi}^{\eta} \sin nut du \\ &= O(1) \frac{\chi^1 \left(\frac{1}{2} \right)}{nt} \end{aligned}$$

and for $\dfrac{1}{2} < \xi^{'} < 1 - h$

$$\begin{aligned} M_{22} &= \int_{\frac{1}{2}}^{1-h} \chi^1(u) R^n(u,t) \sin nut du \\ &= \chi^1(1-h) R^n(1-h, t) \int_{\xi^1}^{1-h} \sin nut du \\ &= O(1) \frac{\chi^1(1-h)}{nt} = O(1) \left(\frac{\chi^1 \left(1 - \frac{\pi}{nt} \right)}{nt} \right) \end{aligned}$$

(since $\chi^1 \uparrow$, $R^n(u,t) \downarrow$ for $0 < u < \frac{1}{2}$ and $\chi^1 \uparrow$, $R^n(u,t) \uparrow$ for $\frac{1}{2} < u < 1$)

Hence collecting the estimates for M_{21} and M_{22} we obtain

$$M_{21} + M_{22} = O(1) \left(\frac{\chi^1 \left(1 - \frac{\pi}{nt} \right)}{nt} \right)$$

Case II For $h = \dfrac{\pi}{nt} > \dfrac{1}{2}$, we have the single integral

$$\begin{aligned} M_2 &= \int_h^{1-h} \chi^1(u) R^n(u,t) \sin nutdu \\ &= O(1)\left(\frac{\chi^1\left(1-\frac{\pi}{nt}\right)}{nt}\right) \end{aligned}$$

From case (I and II) , we get

$$M_2 = O(1)\left(\frac{\chi^1\left(1-\frac{\pi}{nt}\right)}{nt}\right)$$

Collecting the estimates for M_1, M_3 and M_2 we obtain the result .

We omit the proof of Lemma 11(ii) as it can be proved by using arguments similar to those used in proving lemma 11(i) .

4. Proof of the Theorem

Using Lemma 1 and (1.27) we have

$$\begin{aligned} \|l_n(x) - l_n(x+y)\|_p &\le \int_0^\pi \|\varphi_x(t) - \varphi_{x+y}(t)\|_p||dt \\ &= \int_0^\pi \|F(t)\|_p |H_n(t)|\ dt \\ &= \left[\int_0^{\pi/n} + \int_{\pi/n}^\delta + \int_\delta^\pi\right] \|F(t)\|_p |H_n(t)|\ dt \\ &= I + J + K \quad \text{(say)} \end{aligned} \tag{35}$$

Using Lemma 3 and Lemma 4 (iii) , we get

$$\begin{aligned} I &= O(n).|y|^\beta \int_0^{\pi/n} t^{\alpha-\beta}.dt \\ &= O(1)\frac{|y|^\beta}{n^{\alpha-\beta}} \end{aligned} \tag{36}$$

Since $\chi(u)$ is absolutely continuous over $(0,1)$ and χ^1, is positive, using Lemma 3, Lemma 4 (iv) and Lemma 10 we get

$$\begin{aligned}
K &= O(1)|y|^{\beta}\int_{\delta}^{\pi}\frac{t^{\alpha-\beta}}{t}\left\{\int_0^1 e^{-cnu(1-u)t^2}d\chi(u)\right\}dt \\
&= O(1)|y|^{\beta}\int_0^1 \chi^1(u)du\left\{\int_{\delta}^{\pi}\frac{t^{\alpha-\beta}}{t}e^{-cnu(1-u)t^2}dt\right\} \\
&= O(1)|y|^{\beta}\int_0^1 \chi^{'}(u).e^{-cnu(1-u)\delta^2}du \\
&= O(1)|y|^{\beta}\left[\chi\left(\frac{1}{n}\right)+\left\{\chi(1)-\chi\left(1-\frac{1}{n}\right)\right\}\right] \\
&+ O(1)|y|^{\beta}\frac{1}{n^{\alpha-\beta}}\int_{\frac{1}{n}}^{1-\frac{1}{n}}\frac{\chi^1(u)du}{[u(1-u)]^{\alpha-\beta}} \qquad (37)
\end{aligned}$$

$$\left[\text{Since } e^{-cnu(1-u)t^2}\downarrow \text{in } t \text{ and } \frac{t^{\alpha-\beta}}{t}\in L(\delta,\pi)\right]$$

Using (1.5) we have

$$\begin{aligned}
J &= \int_{\frac{\pi}{n}}^{\delta}\|F(t)\|_p|Q_n^r(t)+Q_n^i(t)\cot t/2|dt \\
&\le \int_{\frac{\pi}{n}}^{\delta}\|F(t)\|_p|Q_n^r(t)|dt+\int_{\frac{\pi}{n}}^{\delta}\|F(t)\|_p|Q_n^i(t)||\cot t/2|dt \\
&= \frac{1}{2\pi}\|F(t)\|_p\left|\int_0^1 R^n(u,t)\cos n\Theta d\chi(u)\right|dt \\
&+ \frac{1}{2\pi}\int_{\pi/n}^{\delta}\|F(t)\|_p\frac{dt}{|\tan t/2|}\left|\int_0^1 R^n(u,t)\sin n\Theta d\chi(u)\right| \\
&= J_1+J_2 \quad \text{(say)} \qquad (38)
\end{aligned}$$

We write

$$\begin{aligned} J_1 &= \frac{1}{2\pi}\int_{\frac{\pi}{n}}^{\delta} \|F(t)\|_p \left|\int_0^1 R^n(u,t)(\cos n\Theta - \cos nut + \cos nut)d\chi\ (u)\right| dt \\ &= \frac{1}{2\pi}\int_{\frac{\pi}{n}}^{\delta} \|F(t)\|_p \int_0^1 |R^n(u,t)(\cos n\Theta - \cos nut)d\chi\ (u)| \\ &+ \frac{1}{2\pi}\int_{\frac{\pi}{n}}^{\delta} \|F(t)\|_p dt \left|\int_0^1 R^n(u,t)\cos nut d\chi\ (u)\right| \\ &= J_{11} + J_{12} \quad \text{(say)} \end{aligned} \tag{39}$$

Using Lemma 3 , Lemma 4(ii) and Lemma 6(ii), we get

$$J_{11} = O(1)|y|^{\beta}\int_0^1 \chi^1(u)du \int_{\frac{\pi}{n}}^{\delta} t^{\alpha-\beta}.e^{-cnu(1-u)t^2} nu(1-u)t^3.dt$$

whence putting $nu(1-u)t^2 = \theta$ and writing , $U_1 = \pi^2 u(1-u).n^{-1}, U_2 = \delta^2 u(1-u).n$

We get

$$\begin{aligned} J_{11} &= O(1)\frac{|y|^{\beta}}{n^{\frac{2+\alpha-\beta}{2}}}\left[\int_0^1 \frac{\chi^1(u)du}{[u(1-u)]^{\frac{\alpha-\beta+1}{2}}}\int_{U_1}^{U_2}\theta^{\frac{2+\alpha-\beta}{2}}.e^{-c\theta}.d\theta\right] \\ &+ O(1)\frac{|y|^{\beta}}{n^{\frac{2+\alpha-\beta}{2}}}\left[\int_0^{\frac{1}{n}} + \int_{\frac{1}{n}}^{1-\frac{1}{n}} + \int_{1-\frac{1}{n}}^{1}\right]\frac{\chi^1(u)du}{[u(1-u)]^{\frac{2+\alpha-\beta}{2}}}\int_{U_1}^{U_2}\theta^{\frac{2+\alpha-\beta}{2}}.e^{-c\theta}.d\theta \\ &= l_1 + l_2 + l_3 \quad \text{(say)} \end{aligned} \tag{40}$$

Using first estimate of lemma 8 (taking $k = 2$), we get

$$\begin{aligned} l_1 &= O(1)|y|^{\beta}\chi\left(\frac{1}{n}\right) \\ \text{and } l_3 &= O(1)|y|^{\beta}\left\{\chi(1) - \chi\left(1-\frac{1}{n}\right)\right\} \end{aligned} \tag{41}$$

Using second estimate of Lemma 8 (taking $k = 2$), we get

$$l_2 = O(1)\frac{|y|^{\beta}}{n^{\alpha-\beta}}\int_{\frac{1}{n}}^{1-\frac{1}{n}} \frac{\chi^1(u)du}{[u(1-u)]^{\alpha-\beta}} \tag{42}$$

Collecting the estimates for l_1, l_2 and l_3 we obtain

$$J_{11} = O(1)|y|^{\beta}\left[\chi\left(\frac{1}{n}\right) + \left\{\chi(1) - \chi\left(1 - \frac{1}{n}\right)\right\}\right] + O(1)\frac{|y|^{\beta}}{n^{\alpha-\beta}}\int_{\frac{1}{n}}^{1-\frac{1}{n}} \frac{\chi^1(u)du}{[u(1-u)]^{\alpha-\beta}} \tag{43}$$

For $t > \dfrac{\pi}{n}$ we write $h = \dfrac{\pi}{nt}$.

Now we proceed to deal with J_{12} . From (4.5) , using Lemma 3 and lemma 11 (ii) we get

$$J_{12} = O(1)|y|^{\beta}\int_{\frac{\pi}{n}}^{\delta} t^{\alpha-\beta}dt\left[\chi\left(\frac{\pi}{nt}\right) + \left\{\chi(1) - \chi\left(1 - \frac{\pi}{nt}\right)\right\}\right] + \frac{\chi^1\left(1 - \frac{\pi}{nt}\right)}{nt}$$

whence putting $t = \dfrac{\pi}{nz}$ we obtain

$$\begin{aligned}
J_{12} &= O(1)|y|^{\beta}\int_{\frac{\pi}{\delta n}}^{1}\left(\frac{\pi}{nz}\right)^{\alpha-\beta}\left[\chi(z) + \{\chi(1) - \chi(1-z)\} + \chi^1(1-z).z\right].\frac{dz}{nz^2} \\
&= O(1)\frac{|y|^{\beta}}{n^{\alpha-\beta+1}}\left[\int_{\frac{1}{n}}^{1}\frac{\chi(z)dz}{z^{\alpha-\beta+2}} + \int_{\frac{1}{n}}^{1}\frac{\chi(1) - \chi(1-z)}{z^{\alpha-\beta+2}}dz\right] \\
&+ O(1)\frac{|y|^{\beta}}{n^{\alpha-\beta+1}}\int_{\frac{1}{n}}^{1}\frac{\chi^1(1-z)}{z^{\alpha-\beta+1}}dz \\
&= O(1)\frac{|y|^{\beta}}{n^{\alpha-\beta}}\left[\int_{\frac{1}{n}}^{1}\frac{\chi(z)dz}{z^{\alpha-\beta+1}} + \int_{\frac{1}{n}}^{1}\frac{\chi(1) - \chi(1-z)}{z^{\alpha-\beta+1}}dz\right] \\
&+ O(1)\frac{|y|^{\beta}}{n^{\alpha-\beta}}\int_{\frac{1}{n}}^{1}\frac{\chi^1(1-z)}{z^{\alpha-\beta}}\ dz.
\end{aligned} \tag{44}$$

Combining the results from (4.5),(4.9), (4.10) and using lemma 9, we get

$$J_1 = O(1)\frac{|y|^{\beta}}{n^{\alpha-\beta}}\left[\int_{\frac{1}{n}}^{1}\frac{\chi(z)dz}{z^{\alpha-\beta+1}} + \int_{\frac{1}{n}}^{1}\frac{\chi(1) - \chi(1-z)}{z^{\alpha-\beta+1}}dz\right] + O(1)\frac{|y|^{\beta}}{n^{\alpha-\beta}}\int_{\frac{1}{n}}^{1-\frac{1}{n}}\frac{\chi^1(u)du}{[u(1-u)]^{\alpha-\beta}} \tag{45}$$

We now proceed to deal with J_2.

From (4.4) , we write

$$
\begin{aligned}
J_2 &= \frac{1}{2\pi}\int_{\frac{\pi}{n}}^{\delta} \frac{\|F(t)\|_p}{|\tan t/2|}\left|\int_0^1 R^n(u,t)(\sin n\Theta - \sin nut + \sin nut)d\chi(u)\right| dt \\
&\le \frac{1}{2\pi}\int_{\frac{\pi}{n}}^{\delta} \frac{\|F(t)\|_p}{|\tan t/2|}\int_0^1 \{R^n(u,t)\sin\Theta - \sin nut\}\ \ d\chi(u) \\
&+ \frac{1}{2\pi}\int_{\frac{\pi}{n}}^{\delta} \frac{\|F(t)\|_p}{|\tan t/2|}\left|\int_0^1 R^n(u,t)\sin nut d\chi(u)\right| dt \\
&= J_{21} + J_{22} \quad \text{(say)}
\end{aligned}
\tag{46}
$$

Using Lemma 3, Lemma 4(ii) and Lemma 6(i) we get

$$
\begin{aligned}
J_{21} &= O(1)|y|^\beta \int_{\frac{\pi}{n}}^{\delta} t^{\alpha-\beta-1}dt\int_0^1 e^{-cnu(1-u)t^2}.nu(1-u)t^3.\chi^1(u)du \\
&= O(1)|y|^\beta\int_0^1 \chi^1(u)du\int_{\frac{\pi}{n}}^{\delta} t^{\alpha-\beta}.e^{-cnu(1-u)t^2}..nu(1-u)t^3\ dt
\end{aligned}
$$

whence putting $nu(1-u)t^2 = \theta$ and writing

$$U_1 = \pi^2 u(1-u).n^{-1}, U_2 = \delta^2 u(1-u)n,$$

we get

$$
\begin{aligned}
J_{21} &= O(1)\frac{|y|^\beta}{n^{\frac{\alpha-\beta+1}{2}}}\int_0^1 \frac{\chi^1(u)du}{[u(1-u)]^{\frac{\alpha-\beta+1}{2}}}\int_{U_1}^{U_2}\theta^{\frac{\alpha-\beta+1}{2}}.e^{-c\theta}.d\theta \\
&= O(1)\frac{|y|^\beta}{n^{\frac{\alpha-\beta+1}{2}}}\left[\int_0^{\frac{1}{n}} + \int_{\frac{1}{n}}^{1-\frac{1}{n}} + \int_{1-\frac{1}{n}}^1\right]\frac{\chi^1(u)du}{[u(1-u)]^{\frac{\alpha-\beta+1}{2}}}\int_{U_1}^{U_2}\theta^{\frac{\alpha-\beta+1}{2}}.e^{-c\theta}.d\theta \\
&= L_1 + L_2 + L_3 \quad \text{(say)}
\end{aligned}
\tag{47}
$$

Using first estimate of Lemma 8 (taking $k = 2$), we get

$$L_1 = O(1)|y|^\beta \chi\left(\frac{1}{n}\right) \tag{48}$$

$$\text{and } L_3 = O(1)|y|^\beta\left\{\chi(1) - \chi\left(1-\frac{1}{n}\right)\right\}$$

Using the second estimate of lemma 8 (taking $k = 2$), we get

$$L_2 = O(1)\frac{|y|^\beta}{n^{\alpha-\beta}} \int_{\frac{1}{n}}^{1-\frac{1}{n}} \frac{\chi^1(u)du}{[u(1-u)]^{\alpha-\beta}} \tag{49}$$

collecting the estimates for L_1, L_2 and L_3 we obtain from (4.13)

$$\begin{aligned} J_{21} &= O(1)|y|^\beta \left[\chi\left(\frac{1}{n}\right) + \left\{\chi(1) - \chi\left(1 - \frac{1}{n}\right)\right\}\right] \\ &+ O(1)\frac{|y|^\beta}{n^{\alpha-\beta}} \int_{\frac{1}{n}}^{1-\frac{1}{n}} \frac{\chi^1(u)du}{[u(1-u)]^{\alpha-\beta}} \end{aligned} \tag{50}$$

For $t > \dfrac{\pi}{n}$ we write $h = \dfrac{\pi}{nt}$.
We now proceed to deal with J_{22}

From (4.12), using Lemma 3, Lemma 11(i) and putting $t = \pi/nz$ we obtain

$$\begin{aligned} J_{22} &= O(1)|y|^\beta \int_{\frac{\pi}{n}}^{\delta} t^{\alpha-\beta-1}dt \left[\chi\left(\frac{\pi}{nt}\right) + \left\{\chi(1) - \chi\left(1 - \frac{\pi}{nt}\right)\right\} + \frac{\chi^1\left(1-\frac{\pi}{nt}\right)}{nt}\right] \\ &= O(1)|y|^\beta \int_{\frac{\pi}{\delta n}}^{1} \left(\frac{\pi}{nz}\right)^{\alpha-\beta-1} \left[\chi(z) + \{\chi(1) - \chi(1-z)\} + \chi^1(1-z).z\right] \frac{dz}{nz^2} \\ &= O(1)\frac{|y|^\beta}{n^{\alpha-\beta}} \left[\int_{\frac{1}{n}}^{1} \frac{\chi(z)dz}{z^{\alpha-\beta+1}} + \int_{\frac{1}{n}}^{1} \frac{\chi(1)-\chi(1-z)}{z^{\alpha-\beta+1}}dz\right] \\ &+ O(1)\frac{|y|^\beta}{n^{\alpha-\beta}} + \int_{\frac{1}{n}}^{1} \frac{\chi^1(1-z)}{z^{\alpha-\beta}}\, dz \end{aligned} \tag{51}$$

Combining (4.12) , (4.16), (4.17) and using Lemma 9, we get

$$\begin{aligned} J_2 &= O(1)\frac{|y|^\beta}{n^{\alpha-\beta}} \left[\int_{\frac{1}{n}}^{1} \frac{\chi(z)dz}{z^{\alpha-\beta+1}} + \int_{\frac{1}{n}}^{1} \frac{\chi(1)-\chi(1-z)}{z^{\alpha-\beta+1}}\right] \\ &+ O(1)\frac{|y|^\beta}{n^{\alpha-\beta}} \int_{\frac{1}{n}}^{1-\frac{1}{n}} \frac{\chi^1(u)du}{[u(1-u)]^{\alpha-\beta}} \end{aligned} \tag{52}$$

Collecting the estimates for $I, K, J(J_1, J_2)$ using Lemma 9 and (4.1) we get for $0 \leq \beta < \alpha \leq 1$

$$\begin{aligned}
&\|l_n(x) - l_n(x+y)\|_p \\
&= O(1)\frac{|y|^\beta}{n^{\alpha-\beta}}\left[\int_{\frac{1}{n}}^{1}\frac{\chi(z)dz}{z^{\alpha-\beta+1}} + \int_{\frac{1}{n}}^{1}\frac{\chi(1)-\chi(1-z)}{z^{\alpha-\beta+1}}dz\right] \\
&+ O(1)\frac{|y|^\beta}{n^{\alpha-\beta}}\int_{\frac{1}{n}}^{1-\frac{1}{n}}\frac{\chi^1(u)du}{[u(1-u)]^{\alpha-\beta}}
\end{aligned}$$

which further ensures that

$$\begin{aligned}
&= \sup_{y\neq 0}\frac{\|l_n(x) - l_n(x+y)\|_p}{|y|^\beta} \\
&= O(1)\frac{1}{n^{\alpha-\beta}}\left[\int_{\frac{1}{n}}^{1}\frac{\chi(z)dz}{z^{\alpha-\beta+1}} + \int_{\frac{1}{n}}^{1}\frac{\chi(1)-\chi(1-z)}{z^{\alpha-\beta+1}}dz\right] \\
&+ O(1)\frac{1}{n^{\alpha-\beta}}\int_{\frac{1}{n}}^{1-\frac{1}{n}}\frac{\chi^1(u)du}{[u(1-u)]^{\alpha-\beta}}
\end{aligned}$$

Also $\|\varphi_x(t)\|_p = O(t^\alpha)$ by hypothesis proceeding as above we obtain

$$\|l_n(x)\|_p = O(1)\frac{1}{n^\alpha}\left[\int_{\frac{1}{n}}^{1}\frac{\chi(z)dz}{z^{\alpha+1}} + \int_{\frac{1}{n}}^{1}\frac{\chi(1)-\chi(1-z)}{z^{\alpha+1}}dz\right] + O(1)\frac{1}{n^\alpha}\int_{\frac{1}{n}}^{1-\frac{1}{n}}\frac{\chi^1(u)du}{[u(1-u)]^\alpha}$$

Collecting the estimates for $\sup_{y\neq 0}\frac{\|l_n(x) - l_n(x+y)\|_p}{|y|^\beta}$ and $\|l_n(x)\|_p$
We obtain the result (2.1).
This completes the proof of the theorem.

5. Corollaries

In the special case when $\chi(t) = 1-(1-t)^\delta, 0 < \delta \leq 1$ the Hausdorff mean $H_n(f(x), \chi)$ reduces to familiar Cesaro mean $C_n^\delta(f,x)$ of the Fourier series and hence we can verify that the following result follows from our theorem.

Corollary 1 Let $0 \leq \beta < \alpha \leq 1$ and $f \in H(\alpha, p), p \geq 1$. Then

$$\|C_n^\delta(f,x) - f(x)\|_{\beta,p} = O(1) \begin{cases} \frac{1}{n^{\alpha-\beta}}, & 0 < \alpha - \beta < \delta \leq 1 \\ \frac{\log n}{n^{\alpha-\beta}}, & 0 < \alpha - \beta = \delta = 1 \\ \frac{1}{n^\delta}, & 0 < \delta < \alpha - \beta \leq 1 \end{cases}$$

The case $p = \infty$ and $\delta = 1$ includes Theorem A due to Prösdorff [9]. Putting $p = \infty$ and $\beta = 0$ in Corollary 1 we obtain

Corollary 2 ([1], P-301) If $f \in Lip\alpha, 0 < \alpha \leq 1$, then

$$\|C_n^\delta(f,x) - f(x)\|_c = O(1) \begin{cases} \frac{1}{n^\alpha}, & 0 < \alpha < \delta \leq 1 \\ \frac{\log n}{n^\alpha}, & 0 < \alpha = \delta = 1 \\ \frac{1}{n^\delta}, & 0 < \delta < \alpha \leq 1 \end{cases}$$

References

[1] Alexists . G . : *Convergence Problems of orthogonal series* , Pergamon Press, Newyork 1961 .

[2] Chandra, P : *"On the generalised Fejér means in the metric of Hölder space"* Math. Nachar, 109 (1982) , 39-45 .

[3] Das. G, Ghosh, T and Ray.B.K : *Degree of approximation of functions by their Fourier series in the generalised Hölder metric*, Proc. Indian Acad. Sc. Math.Soc 106 (2) (1996) 139-153 .

[4] Hardy, G.H : *Divergent Series*, Oxford 1956.

[5] Hille , E and Tamarkin J.D : *On the summability of Fourier series III*, Mathematics che-Annalen 108 (1933) 525-577 .

[6] Khan, H.H : *On the degree of approximation of functions belonging to class $Lip(\alpha, p)$*. Indian . Jour. Pure and Appl .math 5 (1974) 132-136 .

[7] Mohapatra R.N. and Russel D.C : *Some direct and inverse theorems in approximation of functions* Jour.Anstralian.math.soc.series (A) 34 (1983) 143-154 .

[8] Peter sen, Gordon M : *Regular matrix transformations*, Mc Graw-Hill Publishing company , London, 1966 .

[9] Prösdorff .S ; *Zur Konvergenz des Fourier reifien Hölder Stetiger Funkt ionen* math , Nachar - 69 (1975) 7-14 .

[10] Quade E.S : *Trigonometric approximation in the mean* Duke math. Jour. 3 (1937) 529 -543 .

[11] Rhodes B.E. : *On the degree of approximation of functions belonging to the weighted $(L^p, \xi(t))$ class by Hausdorff means*, Tamkang J.Math 32(4)(2001), 305-314.

[12] Sahaney, B.N and Rao V.V.G : *Error bounds in the approximation of functions.* Bull.Australian Math.Soc. 6 (1972), 11-18 .

[13] Szasz . O : *Gibbs phenomenon for Hausdorff means*, Trans.Amer . Math.Soc. 69 (1950),440-456 .

[14] Zygmund. A : *Trigonometric series*Vol.1 ., Cambridge University Press , Newyork, 1956.