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Study of Exact Real Fixed Point Problems Using Theory of Variational Inequalities

by

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Abstract

The main goal of this paper is to develop the theory of finding n real fixed points of a real polynomial function of degree n using the theory of variational inequalities. The fixed points equations are also reduced to Legendre polynomial equations. Numerical algorithm for finding the fixed points using the theory of variational inequalities is discussed with relaxed step. To support the theorems of the paper, a pair of examples is illustrated. The theory of fixed points for higher degree polynomials is discussed using Frobenius companion matrix.

Key words: Variational inequalities, real fixed point problems, real fixed points, Frobenius companion matrix.

AMS Classification: 65K10, 90C33, 47J30

1. Introduction

Variational inequality problem (VIP) was developed by G. Stampacchia ([7],1964) to study the Signorini contact problem as a general purpose. Later the theory of VIPs became very interesting because it is simple, direct, unified and one of the most efficient frameworks to solve the equilibrium and non-equilibrium problems. In the recent times the theory of variational inequalities have turned out to be very useful application in studying the problems arising in Applied Mathematics, Applied Physics, Engineering, Medical, Finance etc. Some important problems are constrained/unconstrained optimization problems, image processing problems, contact problems, networking problems, obstacle problems, viscocity problems, nash equilibrium problems and many more.

The theory of fixed point problems (FPP) is used to solve the equilibrium problems and non-equilibrium problems. The researchers have also observed that FPP is a particular case of VIP. Various type of iterative methods are developed to solve the variational equalities using fixed point theorems. Chen [1] has proposed an improved two-step extragradient algorithm for pseudomonotone generalized variational inequalities. He has used two projections at each iteration and which allows one to take different stepsize rules.

Nomirovskii et al. [6] have developed a new two-stage method for the approximate solution of variational inequalities with pseudo-monotone and Lipschitz-continuous operators acting in a finite-dimensional linear normed space using the Bregman divergences.

Recently Das [2] has developed the variable step iterative method to find the existence theorem of T-η-invex function by considering it as a F-variational inequality problem. Later Das et. al [2, 5, 3] have studied some other variational inequalities using the variable step iterative method.

In this paper we develop the theory to generate a real algebraic polynomial function of degree n having exact n real fixed point, called n-exact real fixed point function. We obtain the exactly n real fixed points of the n-exact real fixed point function using the theory of variational inequalities.

2. Real algebraic equation

In general, the real fixed point problem is of finding a real point $x \in \mathbb{R}$ of the function $f \in X^*$ satisfying $f(x) = x$. Let $p_n(x)$ be an algebraic polynomial of degree n, $n \geq 0$ and $y_k(x) = x^k$ be an algebraic monomial with degree k for each $k = 0, 1, \cdots, n$. For $x \in \mathbb{R}$ and $n \in \mathbb{W}$, the set of whole numbers, let

$$\mathbb{Y}_n(x) = \left\{ y \in \mathbb{R} : y = y(x) = \sum_{k=0}^{n} a_k y_k(x) = p_n(x), y_k(x) = x^k,\ a_k \in \mathbb{R}, x \in \mathbb{R} \right\}.$$

We denote

$$[\mathbb{Y}_n(x)] = \left\{ [y(x)] = (y_0(x), y_1(x), \cdots, y_n(x)) \in \mathbb{R}^n :, y_k(x) = x^k, 0 \leq k \leq n \right\}.$$

The dual of $[\mathbb{Y}_n(x)]$ is defined by

$$[\mathbb{Y}_n(x)]^* = \left\{ \alpha = (\alpha_0, \alpha_1, \cdots, \alpha_n) \in \mathbb{R}^n : \alpha[y(x)] = \sum_{k=0}^{n} \alpha_k y_k(x) = y(x) \in \mathbb{R}, x \in \mathbb{R} \right\}$$

and its bidual is defined by

$$[\mathbb{Y}_n(x)]^{**} = \left\{ [y] = (y_0, \cdots, y_n) \in \mathbb{R}^n : \alpha \cdot [y(x)] = \sum_{k=0}^{n} \alpha_k y_k(x) = y(x) \in \mathbb{R}, x \in \mathbb{R} \right\}.$$

For our need we consider $\langle g, f \rangle$ evaluates the value of $g \in [\mathbb{Y}_n(x)]^*$ at $f \in [\mathbb{Y}_n(x)]$.

For $K \subset \mathbb{R}$, let $[y] : K \to \mathbb{R}^{n+1}$ be defined by $[y](x) = [y_k(x)]$ where $y_k : K \to \mathbb{R}$ satisfies $y_k(x) = x^k$ for all $k \in \mathbb{N}$. Assume that $A : K \to [\mathbb{Y}_n(x)]^*$ is defined by

$$A(x) = (\alpha_0, \alpha_1, \cdots, \alpha_n) \in [\mathbb{Y}_n(x)]^* = \mathbb{R}^{n+1}$$

for each $x \in K$. As $A(x) : [\mathbb{Y}_n(x)] \to \mathbb{R}$, we have $A(x)(f) \in \mathbb{R}$ for each $f \in [\mathbb{Y}_n(x)] = \mathbb{R}^{n+1}$. $A(x)[y] = \langle A(x), [y(x)] \rangle$ for $[y] \in [\mathbb{Y}_n(x)]$. Now the canonical embedding $J : [\mathbb{Y}_n(x)] \to [\mathbb{Y}_n(x)]^{**}$ (continuous or discontinuous map) defined by

$$J([y(x)])(\alpha) = \langle \alpha, [y(x)] \rangle = [\alpha]\cdot[y(x)] = \sum_{k=0}^{n} \alpha_k y_k(x) = \sum_{k=0}^{n} \alpha_k x^k = y(x) = \langle A(x), [y(x)] \rangle$$

is a homeomorphism for $\alpha_k \in \mathbb{R}, x \in \mathbb{R}$.

2.1. **Exact real FPP.** In order to study the existence of exact real fixed point problem ($ERFPP$), we need to define the concept of exact real fixed point function.

Definition 2.1. A polynomial function $f(x)$, $x \in K \in \mathbb{R}$ is said to a *n-exact real fixed point function*, if it satisfies an algebraic equation of degree n with real coefficients and has exactly n real solutions which are the fixed points of f.

Let $f(x) \in \mathbb{Y}_n(x)$ be arbitray. Let $F_n(x)$ be the collection all real valued functions $f : X \to \mathbb{R}$ having exactly n real fixed points, i.e.,

$$F_n(x) = \{f : X \to \mathbb{R} | f(x) = x \text{ has exactly } n \text{ real solutions}\}.$$

Consider

$$[F_n(x)] = \{f : X \to \mathbb{R} | f(x) \in F_n(x) \cap \mathbb{Y}_n(x), x \in K\}.$$

The problem of exact real FPP ($ERFPP$-n) is to find a function $f(x) \in \mathbb{R}$ for which the equation $f(x) = x$, $x \in K \subset \mathbb{R}$ satisfies a n degree real algebraic equation $p_n(x) = 0$, i.e., to find $f \in [F_n(x)]$ such that

$$\langle A(x), [f(x)] \rangle = -1 + \sum_{k=1}^{n} \alpha_k x^k = 0 \quad (1)$$

for all $[f(x)] \in [\mathbb{Y}_n(x)]$, $x \in \mathbb{R}$.

Theorem 2.2. *For $k \neq 0$, fix $a = -k^2$ and $b^2 > 4k^2$. Assume that $A(x) = (-1, b, a) \in [\mathbb{Y}_2(x)]^* = \mathbb{R}^3$; then* (1) *gives two real fixed point equation of* *degree* 2 *(2-RFPE) with exactly* 2 *real fixed points and exact real FPP functions satisfying the equations*

$$-k^2x^2 + (2k+m)x - 1 = 0 \tag{2a}$$

$$k^2x^2 + (2k+m)x + 1 = 0 \tag{2b}$$

for some $m > 0 \in \mathbb{R}$. *(From the above two equations one can get the exact real fixed point function* $f(x)$ *with exactly two fixed points.)*

Proof. For any $A(x) = (-1, b, a) \in [\mathbb{Y}_2(x)]^* = \mathbb{R}^3$ and $[f(x)] = (1, x, x^2) \in \mathbb{R}^3$, (1) gives the equation

$$ax^2 + bx - 1 = 0 \tag{3}$$

which has the real roots if $b^2 > 4a$. Choose $a = -k^2$, to get $b^2 > 4k^2$, i.e., $b > 2k$ or $b < -2k$, implying $b = 2k + m$ or $b = -2k - m$ for some $m > 0$.

(a) Putting $a = -k^2$ and $b = 2k + m$ in equation (3), we obtain

$$-k^2x^2 + (2k+m)x - 1 = 0.$$

(b) Putting $a = -k^2$ and $b = -(2k + m)$ in equation (3), we obtain

$$k^2x^2 + (2k+m)x + 1 = 0.$$

This completes the proof. □

Theorem 2.3. *For* $A(x) = (-1, c, b, a) \in [\mathbb{Y}_3(x)]^* = \mathbb{R}^4$ *and* $[f(x)] = (1, x, x^2, x^3)$, *the equation* (1) *has a real point solution* $x = \alpha = k^2 > 0$ *for some* $k \neq 0$*; this is a fixed point of the real fixed point function* $f(x)$. *If the function given in the equation* (1) *has the representation*

$$\langle A(x), [f(x)]\rangle = (x - \alpha)(px^2 + qx + r)$$

where $p = \beta^2 > 0$, $r = \gamma^2 \in \mathbb{R}$ *with* $\gamma^2 = 1/\alpha$, $q = 2\beta\gamma + m$ *or* $q = -2\beta\gamma - m$ *for some* $\beta > 0$, $\gamma > 0$ *and* $m > 0$, *then real algebraic fixed point equations of* *degree 3 are*

$$\beta^2 x^3 + (2\beta k^{-1} - k^2 + m)x^2 + \left(k^{-2} - 2k^2\beta k^{-1} - k^2 m\right) x - 1 = 0 \tag{4a}$$

$$\beta^2 x^3 + (-2\beta k^{-1} - k^2 + m)x^2 + \left(k^{-2} + 2k^2\beta k^{-1} - k^2 m\right) x - 1 = 0 \tag{4b}$$

(From the above two equations one can get the exact real fixed point function $f(x)$ with exactly three fixed points.)

Proof. Since $\alpha = k^2$ is a real fixed point of the function $f(x)$ which is a solution of the equation

$$\langle A(x), [f(x)]\rangle = (x - \alpha)(px^2 + qx + r) = 0, \tag{5}$$

implying

$$px^2 + qx + r = 0$$

which gives two real roots if the discriminant $D = q^2 - 4pr > 0$, i.e., $q > 2\beta\gamma$ and $q < -2\beta\gamma$ for $p = \beta^2 > 0$, $r = \gamma^2 \in \mathbb{R}$. Thus $q = 2\beta\gamma + m$ and $q = -2\beta\gamma - m$ for some $m > 0$.

(a) Putting $p = \beta^2 > 0$, $r = \gamma^2 \in \mathbb{R}$ and $q = 2\beta\gamma + m$ in equation (1), we obtain

$$(x - \alpha)(\beta^2 x^2 + (2\beta\gamma + m))x + \gamma^2 = 0 \tag{6}$$

(b) Putting $p = \beta^2 > 0$, $r = \gamma^2 \in \mathbb{R}$ and $q = 2\beta\gamma + m$ in equation (1), we obtain

$$(x - \alpha)(\beta^2 x^2 - (2\beta\gamma + m))x + \gamma^2 = 0. \tag{7}$$

The equations 6 and 7 are the 3-exact real algebraic fixed point equation of degree 3. Thus

(a) for $\alpha = k^2$ and $\gamma^2 = 1/k^2$, $k > 0$, $q = 2\beta\gamma + m$, (6) gives the exact real algebraic fixed point equation with degree 3 as

$$\beta^2 x^3 + (2\beta k^{-1} - k^2 + m)x^2 + \left(k^{-2} - 2k\beta - k^2 m\right) x - 1 = 0, \tag{8}$$

(b) for $\alpha = k^2$ and $\gamma^2 = 1/k^2$, $k > 0$, $q = -(2\beta\gamma + m)$, (7) gives the exact real algebraic fixed point equation with degree 3 as

$$\beta^2 x^3 + (-2\beta k^{-1} - k^2 + m)x^2 + \left(k^{-2} + 2k\beta - k^2 m\right) x - 1 = 0 \tag{9}$$

which completes the proof of the theorem. □

2.2. **Examples.** Theorem 2.2 and Theorem 2.3 are illustrated by the following examples:

Example 2.4.

(i) For $m = 1$, $k = 1$, equation (2a) gives the real fixed point equation of degree 2 as

$$-x^2 + 3x - 1 = 0$$

which has exactly two real fixed points as $(3 \pm \sqrt{5})/2$. From the above equation, the exact fixed point function $f(x)$ can be obtained.

(ii) For $m = 1$, $k = 2$, equation (2a) gives the real fixed point equation of degree 2 as

$$-4x^2 + 5x - 1 = 0$$

which has exactly two real fixed points as $x_1 = 0.25$ and $x_2 = 1$. From the above equation, the exact fixed point function $f(x)$ can be obtained.

Example 2.5. For $m = 1$ choose $\alpha = \beta = 1$; we get $\gamma = 1$, $q = 3$, $p = 1$ and $r = 1$. Putting the value in the equation (4a), we get

$$(x - 1)(x^2 + 3x + 1) = 0 \Rightarrow x^3 + 2x^2 - 2x - 1 = 0$$

which is the exact real algebraic fixed point equation with degree 3 and has exactly three real solutions as $x_1 = 1, x_2 = -2.618033989$ and $x_3 = -0.3819660113$ which are the fixed point of some function $f(x)$.

3. Study using Legendre Polynomials

Let $\mathcal{L}$ be the set of all Legendre polynomials and $P_k(x) \in \mathcal{L}$ be the Legendre polynomial of degree k, $b_k \in \mathbb{R}$ for each k and

$$\mathbb{P}_n(x) = \left\{ y \in \mathbb{R} : y = w(x) = \sum_{k=0}^{n} b_k P_k(x) \right\}.$$

Each $y \in \mathbb{Y}_n(x)$ can be expressed as

$$y = p_n(x) = \sum_{k=0}^{n} a_k x^k = \sum_{k=0}^{n} b_k P_k(x) \in \mathbb{P}_n(x)$$

for some $b_k \in \mathbb{R}$, $k = 0, 1, \cdots, n$, implying the basis for both the polynomial spaces $\mathbb{Y}_n(x)$ and $\mathbb{P}_n(x)$ are same spaces. and

$$[\mathbb{P}_n(x)] = \left\{ [w(x)] = (P_0(x), P_1(x), \cdots, P_n(x)) \in \mathbb{R}^{n+1} :, P_k(x) \in \mathcal{L}, 0 \leq k \leq n \right\},$$

then $[\mathbb{Y}_n(x)]$ and $[\mathbb{P}_n]$ are equivalence spaces. The dual of $[\mathbb{P}_n(x)]$ is defined by

$$[\mathbb{P}_n(x)]^* = \left\{ \beta = (\beta_0, \beta_1, \cdots, \beta_n) \in \mathbb{R}^{n+1} : \beta[w(x)] = \sum_{k=0}^{n} \beta_k P_k(x) = y(x) \right\}$$

and its bidual is defined by

$$[\mathbb{P}_n(x)]^{**} = \left\{ [w] = (P_0(x), \cdots, P_n(x)) \in \mathbb{R}^n : \beta \cdot [w(x)] = \sum_{k=0}^{n} \beta_k P_k(x) = y(x) \right\}$$

for all $x \in \mathbb{R}$. We consider the pairing $\langle h, w \rangle_*$ that evaluates the value of $h \in [\mathbb{P}_n(x)]^*$ at $f \in [\mathbb{P}_n(x)]$. Assume that the mapping $B : \mathbb{R} \to [\mathbb{P}_n(x)]^*$ is defined by

$$B(x) = (\beta_0, \beta_1, \cdots, \beta_n) \in [\mathbb{P}_n(x)]^* = \mathbb{R}^{n+1}$$

for each $x \in K$. Now the canonical embedding $J_1 : [\mathbb{P}_n(x)] \to [\mathbb{P}_n(x)]^{**}$ (continuous or discontinuous map) defined by

$$J_1([w(x)])(\beta) = \sum_{k=0}^{n} \beta_k P_k(x) = y(x) = \langle B(x), [w(x)] \rangle_*$$

is a homeomorphism for $\beta_k \in \mathbb{R}, x \in \mathbb{R}$. Hence we obtain the relation

$$\begin{array}{ccc} \langle A(x), [y(x)] \rangle & = & \sum_{k=0}^{n} \alpha_k y_k(x) \\ & \shortparallel & \\ & y(x) & \\ & \shortparallel & \\ \sum_{k=0}^{n} \beta_k P_k(x) & = & \langle B(x), [w(x)] \rangle_* \end{array}$$

Hence the Legendre polynomial equation for n-exact fixed point is

$$\langle B(x), [w(x)] \rangle_* = \sum_{k=0}^{n} \beta_k P_k(x) = 0 \tag{10}$$

for each $x \in \mathbb{R}$.

Theorem 3.1. *For $k \neq 0$ and $m > 0$, assume that* $B(x) = \left(\frac{k^2}{3}, 2k \mp m, -\frac{2k^2}{3}\right) \in [\mathbb{P}_2(x)]^* = \mathbb{R}^3; [B(x)] = (P_0(x), P_1(x), P_2(x))$, *then* (10) *gives two real fixed point equation of degree* 2 *(2-RFPE) with exactly* 2 *real fixed points and exact real FPP functions satisfying the equations*

$$-\frac{2k^2}{3} P_2(x) + (2k+m) P_1(x) - \left(1 + \frac{k^2}{3}\right) P_0(x) = 0 \tag{11a}$$

$$\frac{2k^2}{3} P_2(x) + (2k+m) P_1(x) + \left(1 + \frac{k^2}{3}\right) P_0(x) = 0 \tag{11b}$$

for some $m > 0 \in \mathbb{R}$. From the above two equations one can get the exact real fixed point function $f(x)$ with exactly two fixed points.

Proof. By using Theorem 2.2, we have $A(x) = (-1, b, a) \in [\mathbb{Y}_2(x)]^* = \mathbb{R}^3$, the basis element is $[f(x)] = (1, x, x^2) \in \mathbb{R}^3$ associates to the 2-exact fixed point equations as

$$-k^2x^2 + (2k+m)x - 1 = 0$$

$$k^2x^2 + (2k+m)x + 1 = 0.$$

According to Theorem 2.2, we have

(a) for $a = -k^2$ and $b = 2k + m$, equation (3) gives

$$-k^2x^2 + (2k+m)x - 1 = 0,$$

implying

$$-\frac{2k^2}{3}P_2(x) + (2k+m)P_1(x) - \left(1 + \frac{k^2}{3}\right)P_0(x) = 0,$$

i.e.,

$$\langle B(x), [w(x)]\rangle_* = 0$$

where $B(x) = \left(-\left(1 + \frac{k^2}{3}\right), 2k+m, -\frac{2k^2}{3}\right)$ and $[w(x)] = (P_0(x), P_1(x), P_2(x))$;

(b) for $a = -k^2$ and $b = -(2k+m)$, equation (3) gives

$$k^2x^2 + (2k+m)x + 1 = 0,$$

implying

$$\frac{2k^2}{3}P_2(x) + (2k+m)P_1(x) + \left(1 + \frac{k^2}{3}\right)P_0(x) = 0$$

where $B(x) = \left(1 + \frac{k^2}{3}, (2k+m), \frac{2k^2}{3}\right)$ and $[w(x)] = (P_0(x), P_1(x), P_2(x))$.

This completes the proof. $\square$

Theorem 3.2. *For some $k \neq 0$, $m > 0$ and $\beta \in \mathbb{Z}_+$, assume that*

$$y(x) = \langle A(x), [y(x)]\rangle = -1 + a_1x + a_2x^2 + a_3x^3$$

where $A(x) = (-1, a_1, a_2, a_3)$, $a_1 = k^{-2} \mp 2k^2\beta k^{-1} - k^2m$, $a_2 = \pm 2\beta k^{-1} - k^2 + m$ and $a_3 = \beta^2$, $\beta \in \mathbb{Z}_+$, the positive integer set. Then

$$B(x) = (\beta_0, \beta_1, \beta_2, \beta_3) \in [\mathbb{P}_3(x)]^* = \mathbb{R}^4$$

and $[w(x)] = (P_0(x), P_1(x), P_2(x), P_3(x))$, the real algebraic fixed point equations (10) of degree 3 are

$$y = w(x) = \langle B(x), [w(x)]\rangle = \beta_0P_0(x) + \beta_1P_1(x) + \beta_2P_2(x) + \beta_3P_3(x) = 0$$

where $\beta_0 = -1 + \frac{a_2}{3}, \beta_1 = a_1 + \frac{3a_3}{5}, \beta_2 = \frac{2a_2}{3}$ and $\beta_3 = \frac{2a_3}{5}$.

Proof. Given for some $k \neq 0$, $m > 0$ and $\beta \in \mathbb{Z}_+$, assume that

$$y(x) = \langle A(x), [y(x)] \rangle = -1 + a_1 x + a_2 x^2 + a_3 x^3$$

where $A(x) = (-1, a_1, a_2, a_3)$, $a_1 = k^{-2} \mp 2k^2 \beta k^{-1} - k^2 m$, $a_2 = \pm 2\beta k^{-1} - k^2 + m$ and $a_3 = \beta^2$, i.e.,

(1) for $a_1 = k^{-2} - 2k^2 \beta k^{-1} - k^2 m$, $a_2 = 2\beta k^{-1} - k^2 + m$ and $a_3 = \beta^2$, the 3-exact real fixed point equation of degree 3 is

$$y(x) = -1 + a_1 x + a_2 x^2 + a_3 x^3$$

which can be written as

$$y = w(x) = \beta_0 P_0(x) + \beta_1 P_1(x) + \beta_2 P_2(x) + \beta_3 P_3(x) = 0$$

where $\beta_0 = -1 + \frac{a_2}{3}, \beta_1 = a_1 + \frac{3a_3}{5}, \beta_2 = \frac{2a_2}{3}$ and $\beta_3 = \frac{2a_3}{5}$.

(2) For $a_1 = k^{-2} + 2k^2 \beta k^{-1} - k^2 m$, $a_2 = -2\beta k^{-1} - k^2 + m$ and $a_3 = \beta^2$, the 3-exact real fixed point equation of degree 3 is

$$y(x) = -1 + a_1 x + a_2 x^2 + a_3 x^3$$

which can be written as

$$y = w(x) = \beta_0 P_0(x) + \beta_1 P_1(x) + \beta_2 P_2(x) + \beta_3 P_3(x) = 0$$

where $\beta_0 = -1 + \frac{a_2}{3}, \beta_1 = a_1 + \frac{3a_3}{5}, \beta_2 = \frac{2a_2}{3}$ and $\beta_3 = \frac{2a_3}{5}$.

which completes the proof of the theorem. □

4. Fixed point using VIP

Let $X = \mathbb{R}$ and $K \subset X$ be a nonempty convex compact subset of X. Let $f : K \to \mathbb{R}$ be an n-exact real fixed point function, i.e., $f \in [F_n(x)]$. By fixed point theory, equation (1) can be written as

$$x = F(x)$$

where $F(x) = x - \rho \langle A(x), [f(x)] \rangle$ satisfying

$$\left| \frac{dF}{dx} \right| = \left| \frac{d}{dx} \left(x - \rho \langle Ax, [f(x)] \rangle \right) \right| < 1$$

for all $x \in [a, b]$ with the condition $\langle A(a), [f(a)]\rangle \langle A(b), [f(b)]\rangle < 0$.

The iterative method is

$$x_{n+1} = F(x_n), n = 0, 1, \cdots,$$

i.e.,

$$\langle x_{n+1} - x_n, v - x_n\rangle + \rho_n \langle A_n(x), v - x_n\rangle \geq 0$$
$$A_n(x) = \langle A(x_n), [f(x_n)]\rangle$$

for $n = 0, 1, 2, \cdots$ and for all $v \in K$ which is the numerical method for variational inequality problem to find the fixed point of the real function $f(x)$ with real coefficients where the step length is

$$\rho_n \leq -\min_{v\in X} \frac{\langle x_{n+1} - x_n, v - x_n\rangle}{\langle A_n(x), v - x_n\rangle}$$
$$A_n(x) = \langle A(x_n), [f(x_n)]\rangle$$

for each step $n = 0, 1, 2, \cdots$.

4.1. **Numerical Method for the Examples 2.4 and 2.5.**

(1) In Example 2.4(i), if we take $f(x) = -x^2 + 3x - 1$

then we get two real fixed points x_1 and x_2 in the interval $[0, 3]$. To use variational inequality problem, we have $F(x) = x - f(x) = x^2 - 2x + 1$ satisfying the condition $|F'(x)| = |2x - 2| < 1$, i.e., $|x - 1| < 1/2$, i.e., $0.5 < x < 1.5$ which is valid. The step wise algorithm for the numerical variational equality method is

(a) Choose $\rho_0 > 0$, any $x_0 \in [0, 1] = K_1$ for x_1 and $x_0 \in [2, 3] = K_2$ for x_2,

(b) for $n = 0, 1, 2, \cdots$, compute

$$\begin{aligned} x_{n+1} &= F(x_n) \\ \rho_n &= -\min_{v\in X} \frac{(x_{n+1} - x_n)(v - x_n)}{(-x_n^2 + 3x_n - 1)(v - x_n)} \text{ for all } v \in K_i, i = 1, 2. \\ x_n &= x_n + \rho_n \end{aligned}$$

(2) In Example 2.5, if we take $f(x) = x^3 + 2x^2 - 2x - 1$ then $f(-3)f(-2) < 0$, $f(-1)f(0) < 0$ and $f(0)f(2) < 0$, so there are three real fixed points x_1, x_2 and x_3 in the interval $[-3, 2]$. To use variational inequality problem, we have $F(x) = x - f(x) = -x^3 - 2x^2 + 3x + 1$ satisfying the condition $|F'(x)| = \left|-3x^2 - 4x + 3\right| < 1$, i.e., $-1 < -3x^2 - 4x + 3 < 1$. Algorithm for the numerical variational equality method, the step wise algorithm is

(a) Choose $\rho_0 > 0$, any $x_0 \in [-3, -2] = K_1$ for x_1, $x_0 \in [-1, 0] = K_2$ for x_2 and $x_0 \in [0, 2] = K_3$ for x_2

(b) for $n = 0, 1, 2, \cdots$, compute

$$\begin{aligned} x_{n+1} &= F(x_n) \\ \rho_n &= -\min_{v \in X} \frac{(x_{n+1} - x_n)(v - x_n)}{(x_n^3 + 2x_n^2 - 2x_n - 1)(v - x_n)} \text{ for all } v \in K_i, i = 1, 2. \\ x_n &= x_n + \rho_n \end{aligned}$$

4.2. **Companion Matrix and Exact real fixed point function of higher degree $n \geq 3$.** Since $f(x)$ is a real algebraic polynomial of degree n, it can be written as

$$f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = p_n(x)$$

where a_i's are real numbers. The polynomial function $p_n(x)$ can be expressed associated with the real Frobenius companion matrix [4] A defined by

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{pmatrix}.$$

Thus the characteristic polynomial $\det(A - xI)$ is $\pm p_n(x)$ ([4]), i.e.,

$$p_n(x) = \begin{vmatrix} x & -1 & 0 & \cdots & 0 \\ 0 & x & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & -1 \\ -a_0 & -a_1 & -a_2 & \cdots & x + a_{n-1} \end{vmatrix}.$$

Since x is the eigen value of A, then x satisfies the polynomial equation $p_n(x) = 0$ (Mora [4]). Now the companion matrix A is diagonalizable, so the diagonal elements of the eigen values of A which are the roots of $p_n(x)$. Hence these eigen values are fixed points of the n-exact real fixed point function $f(x)$. If y is the eigen vector corresponding to eigen value x of A, then the eigen values of the companion matrix can be obtained using power method, i.e.,

$$y_{n+1} = Ay_n = x_n y_n, \ n = 0, 1, 2, \cdots,$$

where x_n is the eigen value of the matrix A in $(n+1)^{\text{th}}$ step. The above numerical method can be expressed as a numerical method of a variational inequality problem to find $y_n \in V$, a closed convex subset in $\mathbb{R}^n$ and $x_n \in \mathbb{R}$ satisfying

$$\langle y_{n+1} - y_n, v - y_n \rangle + \rho_n \langle F(y_n), z - y_n \rangle \geq 0$$

and $F(y_n) = (I - A)y_n$ for $n = 0, 1, 2, \cdots$ for all $z \in V$. The iterative step is

$$\begin{cases} n = 0, 1, 2, \cdots \\ y_{n+1} = Ay_n = x_n y_n^* \\ y_n = y_n * \\ F(y_n) = (I - A)y_n \\ \rho_n = -\min\limits_{z \in V} \frac{\langle y_{n+1} - y_n, z - y_n \rangle}{\langle F(y_n), z - y_n \rangle} \\ x_{n+1} = x_n + \rho_n, f(x_{n+1}) = p_n(x_n) \end{cases}$$

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